APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Spring 2017 (January)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: January 25th, 2017

Time: 9:00 AM – 1:00 PM

A1. Consider the conservation law

$$u_t + (u^4)_x = 0,$$

(a). Find the solution at $t = 1$ with the following initial condition:

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & 0 \le x \le 2 \\ 0 & x > 2 \end{cases}.$$

(b). Solve the Riemann problem (You must consider both $u_l > u_r$ and $u_l < u_r$):

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}.$$

(c). Find the Riemann solution at $x/t = 0$.

A2. For the heat equation

$$u_t = \nu u_{xx}, \quad (x,t) \in \Omega : (-\infty, \infty) \times (0, \infty),$$

when ν is a constant, the solution can be found via Fourier transform and in the integral form as

$$u(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} g(y) e^{-(x-y)^2/4\nu t} dy.$$

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(a). Find the solution with $\nu = 0.25$ and the initial condition

$$u(x,0) = \delta(x+1).$$

(b). Find the solution with $\nu = 0.25$ and the initial condition

$$u(x,0) = H(x-1),$$

where $H(x)$ is the heaviside function,

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}.$$

Write the solution in terms of error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy.$$

(c). Sketch the solutions of (a) and (b) as t increases: $t \to \infty$.

(d). How to modify the solution so that it can also solve for $\nu = \nu(t)$?

A3. Compute

$$I = \int_0^{\pi} \frac{1}{2 - \cos t} dt.$$

(Hint: First show that

$$I = \frac{1}{2} \int_0^{2\pi} \frac{1}{2 - \cos t} dt.\bigg)$$

A4. Let f be a complex-valued function in the unit disc $D = \{|z| \le 1\}$. Suppose that $g = f^2$ and $h = f^3$ are analytic. Prove that f is analytic as well.

A5. Prove for any $a \in \mathbf{C}$ and $n \geq 2$ that the polynomial $az^2 + z + 1$ has at least one root in the disc $\{|z| \leq 2\}$. (Hint: Consider the cases $|a| < \frac{1}{2^n}$ and $|a| \geq \frac{1}{2^n}$ separately.)

B6. Suppose we are given n samples (t_i, y_i), $i = 1, \ldots, n$.

a) (5 points) Give the Lagrange polynomial basis $\ell_j(t)$ for $j = 1, \ldots, n$. Express the polynomial interpolation in terms of these basis.

b) (5 points) Explain how to evaluate the Lagrange polynomial interpolant in $\mathcal{O}(n)$ operations, after some preprocessing step with $\mathcal{O}(n^2)$ operations.

B7. Given the two-point boundary value problem

$$\begin{aligned} -u'' + 2u &= f(x),\ 0 \le x \le 1, \\ u(0) &= a \\ u(1) &= b \end{aligned}$$

a) (5 points) Set up a finite difference approximation for this problem with n equidistant interior points. Describe the resulting linear system.

b) (5 points) Set up a Galerkin method for this problem, based on piecewise linear elements (i.e., the "hat" functions) with n equidistant points in the interior. Describe the resulting linear system.

B8.

(a) Derive a finite difference leapfrog scheme for an initial-value problem for one-dimensional heat equation on an infinite interval, $v_t = \nu v_{xx}$, $v(x, 0) = f(x)$.

(b) Perform stability analysis using the discrete Fourier transform.

B9.

(a) Solve the Riemann problem for a linear system of strictly hyperbolic PDE's

$$U_t + AU_x = 0, \quad U(x,0) = \begin{cases} U_l, & x < 0 \\ U_r, & x > 0 \end{cases}.$$

(i.e., express the Riemann problem solution in terms of eigenvalues, eigenvectors, and the jump $[U] = U_r - U_l$.)

(b) Using the Riemann problem solution, derive the Godunov method for this linear system and show that it is equivalent to the upwind method.

B10.

(a) Define monotone methods for nonlinear hyperbolic scalar conservation laws and describe their main properties.

(b) Investigate whether the conservative upwind method for a nonlinear hyperbolic scalar conservation law is monotone.